Eur. Phys. J. D **41**, 547–556 (2007)
DOI: 10.1140/epjd/e2006-00256-2

The European Physical Journal D

Statistical properties for a class of nonlinear squeezed states

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Received 25 October 2005 / Received in final form 20 February 2006
Published online 29 November 2006 –

Abstract. In the present paper a class of nonlinear squeezed vacuum and first order excited squeezed states are introduced. Under conditions on the non-linearity function, the first order excited squeezed states are realizations of the SU(1,1) group. However, when the condition of the nonlinear function being unitary is removed, these states are defined as eigenstates of certain operators. Some of the properties of these states are investigated for the case of trapped ions. The normalized second-order correlation function, the phase properties, the quasiprobability distribution function and the position distribution of the nonlinear squeezed vacuum and first order excited squeezed states are discussed.

PACS. 42.50.-p Quantum optics – 42.50.Dv Nonclassical states of the electromagnetic field, including entangled photon states; quantum state engineering and measurements

1 Introduction

An important concept that emerges from the study of a quantum harmonic oscillator is the notion of coherent states [1–5]. These states defined as eigenstates of the annihilation operator $\hat{a}$ provides us with a link between quantum and classical oscillators. Historically these states corresponding to the Heisenbereg-Weyl algebra were first constructed by Schrödinger [1]. They can be produced by acting on the vacuum states $|0\rangle$ by the Glauber displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, where $\hat{a}^\dagger$ is the reaction operator and α is a complex number and α^* its conjugate [2]. Moreover, a coherent state is a phase coherent superposition with Poissonian distribution of number states. They are a set of minimum uncertainty states, which are as noiseless as the vacuum state. Solutions of the Schrödinger equation for a charge in a magnetic field corresponding to non-spreading wave packets with a classical dynamics-the coherent states in the modern terminology-were first built by Darwin as early as in 1928 [6]. More recently, the coherent states in the magnetic field problem have been extensively studied by Malkin and Manko [7], and Feldman and Kahn [8] (see also Refs. [9–11]). In the coherent states, the quantum uncertainties in the X- and Y-coordinates of the Larmor center are equal. Generalizations of the coherent states appear as nonlinear coherent states [12,13] and f-coherent state [14], which are eigenstates of the operator $\hat{a}f(\hat{n})$ where $f(\hat{n})$ is a function of the number operator $\hat{n}$. The generalization has been extended to include k-photon coherent states by G D'Ariano and coworkers [15–17] and by Netto and coworkers [18]. Some of their properties have been studied and their time evolution under the action of the Hamiltonian of free harmonic oscillator [18] or an harmonic oscillator [19].

Another class of the minimum uncertainty states is the class of squeezed states [20,21]. The squeezed state is the eigenfunction of an operator, $\hat{b}$. This operator is a linear combination of the creation $\hat{a}^\dagger$ and destruction $\hat{a}$ operators of the field. The mixing of the operators is controlled by means of a variable r, which is called the squeezing parameter, $\hat{b} = \hat{a}\cosh(r) + \hat{a}^\dagger \sinh(r)e^{i\phi}$. Thus when $r = 0$ the coherent state results [1–5,20,21]. These states can be produced by acting with the squeezing operator $\hat{S}(z) = \exp(z\hat{a}^{\dagger 2}/2 - z^*\hat{a}^2/2)$ on the coherent state $|\alpha\rangle$, where z is a complex number. Squeezed states have attracted much attention during the past three decades [22]. These states are important because they can achieve lower quantum noise than the zero-point fluctuations of the vacuum or coherent states. Thus they provide a way of manipulating quantum fluctuations and have a promising future in different applications ranging from optical communications to gravitational wave detection [22]. Indeed, squeezed states have been explored in a variety of non-quantum-optics systems, including classical squeezed states [23]. Actually there are a number of suggested, and actual, applications of these states in quantum information processing including: quantum cryptography [24], quantum teleportation [25], dense coding [26] and quantum communication [27] to name but a few. They have also been proposed for high precision measurements such as improving the sensitivity of Ramsey fringe interferometry [28] and the detection of weak tidal forces due to gravitational radiation.

Various generalizations to the squeezed states have been done by Dodonov et al. [29,30] and Aragone [31].

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Superposition of these states can be thought of as eigenstates of higher powers of the operator $\hat{b}$ [32] and this can be thought of an generalized squeezed states. Other types of photon squeezed states have been introduced [15,33] by taking a general type of the squeezing operator depending on the generalized multiphoton Bose operators or as minimum uncertainty states [18]. Some of their properties have been discussed in there papers and also in references [19, 34].

In this paper, the nonlinear vacuum squeezed state (NLVSS) and the nonlinear first order excited squeezed states (NLESS) (Sect. 2) are introduced. In Section 3, the nonlinear squeezing states are constructed when the non-linearity operator valued function $\hat{f}(\hat{N})$ is not a unitary operator. In studying non-classical properties of fildes, it is natural to introduce nonlinear states as recently reviewed [35]. Some of the non-classical properties of the present states are discussed, including the autocorrelation function, normal quadrature squeezing and amplitude-squared squeezing, the phase properties, the quasi-probability distribution function and the position distribution when the non-linearity function describes the motion of trapped ions (Sect. 4). Finally conclusion is given in Section 5.

2 Nonlinear squeezed states as realizations of SU(1,1)

It is well-known that optical effects connected with two-photon physics are often related to the SU(1,1) Lie group [36–40]. It has been shown that the single- and two-mode bosonic realizations of the SU(1,1) Lie groups have immediate relevance to the non-classical squeezing properties of light [36–40]. As it is well-known the SU(1,1) Lie group is spanned by the three generators K_1, K_2 and K_3 which satisfy the following commutation relations

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \tag{1}$$

The raising (lowering) $K_+(K_-)$ operator defined by $K_\pm = K_1 \pm iK_2$ may be used, to cast the commutation relations in the following:

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3. \tag{2}$$

For any irreducible representation of SU(1,1), the Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ has the form $k(k-1)I$. Thus a reducible representation of the group is determined by the Bergmann number k.

The single-mode squeezed vacuum realization of the SU(1,1) group is considered by taking the generators in the form

$$K_- = \frac{1}{2}\hat{a}^2, \quad K_+ = K_-^\dagger = \frac{1}{2}\hat{a}^{\dagger 2}, \quad K_3 = \frac{1}{2}\left(\hat{N} + \frac{1}{2}\right) \tag{3}$$

with $\hat{N} = \hat{a}^\dagger\hat{a}$ the photon number operator. The Casimir operator in this case becomes $k(k-1) = -3/16$. Therefore there are two irreducible representations associated with $k = 1/4$ and $k = 3/4$ [36,38]. The state space associated with $k = 1/4$ is the even Fock sub-space with the orthonormal basis set $\{|2n\rangle\}$, while the space associated with $k = 3/4$ is the odd Fock sub-space $\{|2n+1\rangle\}$. The squeezed operator

$$S(z) = \exp(\xi K_+ - \xi^* K_-) \tag{4}$$

is the unitary group operator for the single-mode two-photon realization with the generators given by equation (3) and $\xi = (z/|z|)\tanh|z| = e^{i\phi}\tanh r$.

The SU(1,1) coherent states are the single-mode squeezed states. For $k = 1/4$, the squeezed vacuum is given by

$$\left|z; \frac{1}{4}\right\rangle = S(z)|0 = (1-|\xi|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!}\left[\frac{\xi}{2}\right]^n |2n\rangle, \tag{5a}$$

and for $k = 3/4$, the first order excited squeezed state is given by

$$\left|z; \frac{3}{4}\right\rangle = S(z)|1\rangle = (1-|\xi|^2)^{\frac{3}{4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{n!}\left[\frac{\xi}{2}\right]^n |2n+1\rangle. \tag{5b}$$

After this brief discussion, we may refer to reference [41] where the nonlinear squeezed states as realization of the SU(1,1) group have been introduced. The K-operators are modified in the following way

$$K_- = \frac{1}{2}(\hat{a}\hat{f}(\hat{N}))^2, \quad K_+ = \frac{1}{2}(\hat{f}^\dagger(\hat{N})\hat{a}^\dagger)^2, \tag{6a}$$

with $\hat{f}(\hat{N})$ an operator valued function of the photon number operator $\hat{N}$. For the operator K_3 to be in the form (3), then the operator valued function $\hat{f}(\hat{N})$ must be a unitary operator, i.e. $\hat{f}^\dagger = \hat{f}^{-1}$. Under this condition of $\hat{f}$ being unitary, we get

$$\begin{aligned} K_3 &= \frac{1}{8}\left[(\hat{a}\hat{f}(\hat{N}))^2(\hat{f}^\dagger(\hat{N})\hat{a}^\dagger)^2 - (\hat{f}^\dagger(\hat{N})\hat{a}^\dagger)^2(\hat{a}\hat{f}(\hat{N}))^2\right] \\ &= \frac{1}{2}\left(\hat{N} + \frac{1}{2}\right). \end{aligned} \tag{6b}$$

Therefore the coherent states of SU(1,1) are the non-linear squeezing states. Consequently the NLVSS is given by

$$\left|z; \frac{1}{4}\right\rangle_f = (1-|\xi|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n![f(2n)]!}\left[\frac{\xi}{2}\right]^n |2n\rangle, \tag{7a}$$

while the NLESS is given by

$$\left|z; \frac{3}{4}\right\rangle_f = (1-|\xi|^2)^{\frac{3}{4}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{n![f(2n+1)]!}\left[\frac{\xi}{2}\right]^n |2n+1\rangle, \tag{7b}$$

where $f(\cdot)$ is a complex valued function with $\bar{f}(\cdot)$ its complex conjugate satisfying $f\bar{f} = 1$. With $[f(n)]! = \prod_{i=0}^n f(i)$ and $[f(0)]! = 1$. The states (7) are the SU(1,1) group realizations by NLVSS and NLESS. Equation (7a) is a generalization of equation (39a) in reference [42].

3 The case of non-unitary $\hat{\mathbf{f}}$

Even when the operator valued function $\hat{f}(\hat{N})$ is not a unitary operator, one can still define non-linear squeezed states [41]. If we have

$$\hat{A} = \hat{a}\hat{f}(\hat{N}), \quad \hat{A}^{\dagger} = \hat{f}^{\dagger}(\hat{N})\hat{a}^{\dagger}. \tag{8a}$$

Then the canonical conjugate operators are

$$\hat{B} = \hat{a}\frac{1}{\hat{f}^{\dagger}(\hat{N})}, \quad \hat{B}^{\dagger} = \frac{1}{\hat{f}(\hat{N})}\hat{a}^{\dagger}. \tag{8b}$$

Then the commutation relations

$$[\hat{A}, \hat{B}^{\dagger}] = 1, \qquad [\hat{B}, \hat{A}^{\dagger}] = 1. \tag{8c}$$

are found, but $[\hat{A}, \hat{B}] \neq 0$.

There is no loss of generality to assume f to be a well behaved real function which we suppose in what follows. To construct non-linear squeezed states of the form (7) when f is not a unitary operator, we look for the eigenstates of the operator

$$\hat{C} = (1 - |\xi_1|^2)^{-1/2}(\hat{A} - \xi_1\hat{B}^{\dagger}) \tag{9}$$

with eigenvalue zero. That is for the NLVSS we look for the solutions of the equation

$$\hat{C}|\Psi\rangle = 0. \tag{10}$$

It is straightforward to find the expression

$$|\Psi\rangle = N\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n![f(2n)]!}\left[\frac{\xi_1}{2}\right]^n|2n\rangle, \tag{11a}$$

with N given from

$$|N|^{-2} = \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2[f(2n)!]^2}\left[\frac{\xi_1}{2}\right]^{2n}. \tag{11b}$$

While the NLESS are the solutions of the eigenvalue equation

$$\hat{C}^2|\Phi\rangle = 0. \tag{12}$$

Carrying out the calculations, it is easy to find that these states are composed only of the odd Fock states and are given by

$$|\Phi\rangle = N'\sum_{n=0}^{\infty}\frac{\sqrt{(2n+1)!}}{n![f(2n+1)]!}\left[\frac{\xi_1}{2}\right]^n|2n+1\rangle, \tag{13a}$$

with N' the normalization constant given by

$$|N'|^{-2} = \sum_{n=0}^{\infty}\frac{(2n+1)!}{(n!)^2[f(2n+1)!]^2}\left[\frac{\xi_1}{2}\right]^{2n}. \tag{13b}$$

The formulae (11) and (13) are generalizations of equations (5) and formally similar to equation (7) which are the squeezed states and non-linear squeezed states realizations of the SU(1,1) group for the different Bergmann numbers. However, the non-linearity function $f(n)$ is no longer a unitary operator as in Section 2. For the states (11) and (13) to exist then the nonlinearity function $f(n)$ has to be chosen such that the normalizing constant N and N' must be bound.

We may look at the functions $|\Psi\rangle$ and $|\Phi\rangle$ as results of application of exponential operators on the states $|0\rangle$ and $|1\rangle$. In effect we have

$$|\Psi\rangle = N\exp\left[\frac{1}{2}\xi_1 B^{\dagger 2}\right]|0\rangle, \;\; |\Phi\rangle = N'\exp\left[\frac{1}{2}\xi_1 B^{\dagger 2}\right]|1\rangle, \tag{14}$$

where $B^{\dagger}$ is given by (8b). In what follows we discuss some of the statistical properties of the NLVSS and NLESS. Generation of nonlinear states with arbitrary nonlinearity functions can be engineered using a number of laser fields in trapped ions experiments for the quantized states of the atomic centre-of-mass motion [43].

4 Non-classical properties

In this section we shall study the non-classical properties such as, sub-Poissonian effect, normal quadrature squeezing, amplitude squared squeezing, quasi-probability distribution functions, quadrature distribution function and the phase probability distribution function for the NLVSS $|\Psi\rangle$ of equations (11) and of the NLESS $|\Phi\rangle$ of equations (13). As it has been just mentioned at the end of the last section arbitrary nonlinearity functions can be tailored at well to produce nonlinear states [43]. In the present work the nonlinearity function will be taken as $f(\hat{n}) = \sqrt{\hat{n}}$ and $f(0) = 1$ [44]. A choice of this type appears in Hamiltonians describing interaction with intensity-dependent coupling between a two level atom and the electromagnetic field [45]. In contrast to the study of [19,46], we calculate the expectation values of the operators $\hat{a}$ and $\hat{a}^{\dagger}$ and their powers instead of the deformed operators. In what follows some of the nonclassical effects will be discussed.

4.1 Sub-Poissonian distribution

Now, we shall employ the Glauber second-order autocorrelation function to study the sub-Poissonian effect on the present states given by equations (11) and (13) and we take $\xi = e^{i\phi}\tanh r$ with $\phi = \pi$. The Glauber second-order correlation function $g^{(2)}(0)$ is defined by:

$$g^{(2)}(0) = \frac{\langle a^{\dagger 2}a^2\rangle}{\langle a^{\dagger}a\rangle^2} = \frac{\langle \hat{n}^2\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2}. \tag{15}$$

For the choice $f(n) = \sqrt{n}$ then,

$$|N| = |N'| = \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left[\frac{\tanh r}{2}\right]^{2n}\right]^{-\frac{1}{2}}.$$

The expectation value of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ for the NLVSS (11), is given by

$$\langle\hat{n}\rangle_V = |N|^2 \sum_{n=0}^{\infty}(2n)\frac{1}{(n!)^2}\left[\frac{\tanh r}{2}\right]^{2n}, \tag{16a}$$

and for the NLESS (13), it is given by

$$\begin{aligned}\langle\hat{n}\rangle_E &= |N|^2 \sum_{n=0}^{\infty}(2n+1)\frac{1}{(n!)^2}\left[\frac{\tanh r}{2}\right]^{2n}\\ &= \langle\hat{n}\rangle_V + 1.\end{aligned} \tag{16b}$$

While the expectation value of $\hat{n}^2$ for the NLVSS is given by

$$\langle\hat{n}^2\rangle_V = |N|^2 \sum_{n=0}^{\infty}(2n)^2\frac{1}{(n!)^2}\left[\frac{\tanh r}{2}\right]^{2n}, \tag{17a}$$

and for the NLESS is given by

$$\begin{aligned}\langle\hat{n}^2\rangle_E &= |N|^2 \sum_{n=0}^{\infty}(2n+1)^2\frac{1}{(n!)^2}\left[\frac{\tanh r}{2}\right]^{2n}\\ &= \langle\hat{n}^2\rangle_V + 2\langle\hat{n}\rangle_V + 1.\end{aligned} \tag{17b}$$

A light field has a sub-Poissonian distribution if $g^{(2)}$ $(0) < 1$ which is a nonclassical effect, super-Poissonian distribution if $g^{(2)}(0) > 1$, which is a classical effect and Poissonian distribution (characteristic of the coherent state) if $g^{(2)}(0) = 1$.

In Figure 1a, the plot for $g^{(2)}(0)$ of the NLVSS exhibits super-Poissonian behaviour for the distribution. But in Figure 1b, the plot for $g^{(2)}(0)$ of the NLESS exhibits sub-Poissonian behaviour for the distribution for small values of r. Compare the relation between the expectation values of the operators in the NLVSS and their counter parts in the NLESS as it is stated in equations (16b) and (17b). But as the value of r starts to increase we find that the state shows super-Poissonian behaviour as $r > 0.3$.

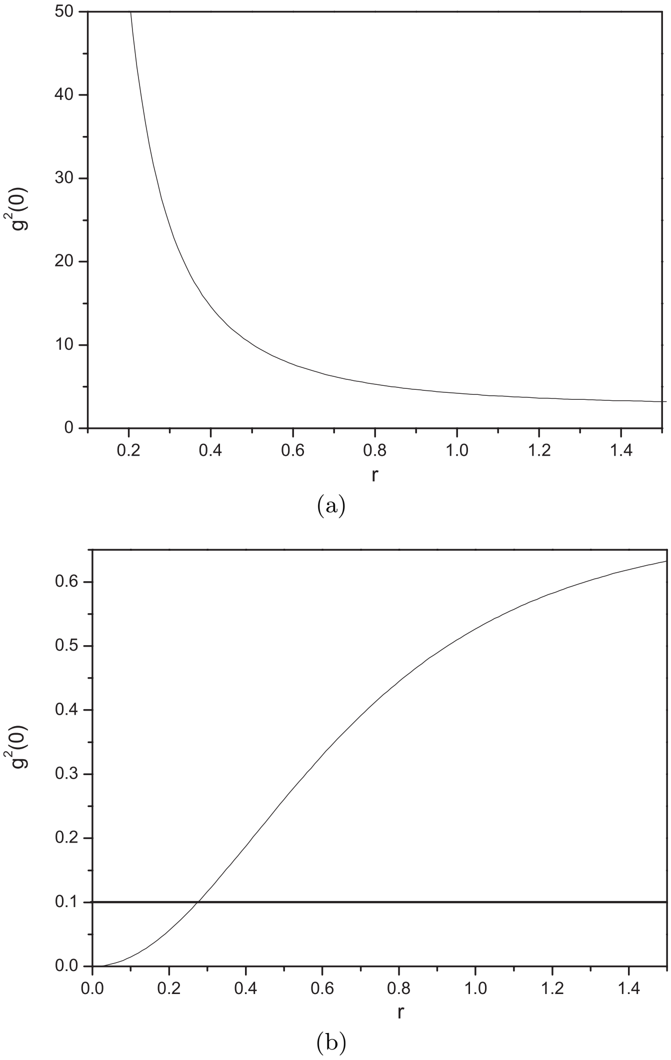


Fig. 1. Autocorrelation function of the NLVSS (a) and for the NLESS (b).

4.2 Normal quadrature squeezing (NQS)

The investigation of NQS is based on defining the two field quadratures

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger). \tag{18}$$

These quadratures satisfy the commutation relation

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \tag{19}$$

Therefore the uncertainty relation for $\hat{X}_1$ and $\hat{X}_2$ is given by

$$(\Delta\hat{X}_1)^2(\Delta\hat{X}_2)^2 \geq \frac{1}{16}. \tag{20}$$

where the quadrature variances is defined

$$(\Delta\hat{X}_i)^2 = \langle\hat{X}_i^{\,2}\rangle - \langle\hat{X}_i\rangle^2, \quad i = 1, 2. \tag{21}$$

The field is said to be squeezed if $\Delta\hat{X}_i^2 \leq 1/4$ ($i = 1$ or 2). Then NQS holds if the squeezing parameters $V(x_1)$ and $V(x_2)$ satisfy the following conditions

$$\begin{aligned}V(X_1) &= 2(\Delta\hat{X}_1)^2 - \frac{1}{2}\\ &= \mathrm{Re}\langle\hat{a}^2\rangle + \langle\hat{n}\rangle - 2(\mathrm{Re}\langle\hat{a}\rangle)^2 < 0,\end{aligned} \tag{22}$$

$$\begin{aligned}V(X_2) &= 2(\Delta\hat{X}_2)^2 - \frac{1}{2}\\ &= \langle\hat{n}\rangle - \mathrm{Re}\langle\hat{a}^2\rangle + 2(\mathrm{Im}\langle\hat{a}\rangle)^2 < 0.\end{aligned} \tag{23}$$

The squeezing parameters depend on the expectation values of the creation and annihilation operator ($\hat{a}^\dagger$ and $\hat{a}$) and their powers. From equations (11) for NLVSS, the

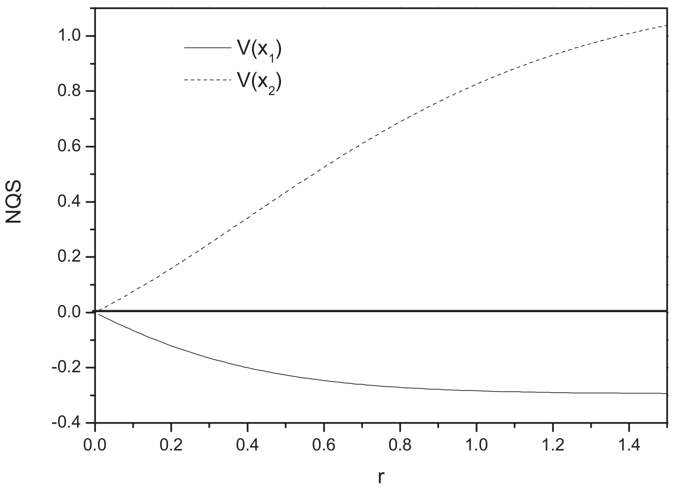


Fig. 2. Normal squeezing of the NLVSS.

expectation values of the operators $\hat{a}$ and $\hat{a}^2$ are

$$\langle \hat{a} \rangle_V = 0, \tag{24a}$$

$$\langle \hat{a}^2 \rangle_V = |N|^2 \left[-\frac{\tanh r}{2} \right] \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)(2n+2)}}{(n!)(n+1)!} \left[\frac{\tanh r}{2} \right]^{2n}. \tag{25a}$$

Also from equation (13) for NLVSS, the expectation values of the operators $\hat{a}$ and $\hat{a}^2$ are

$$\langle \hat{a} \rangle_E = 0, \tag{24b}$$

$$\langle \hat{a}^2 \rangle_E = |N|^2 \left[-\frac{\tanh r}{2} \right] \sum_{n=0}^{\infty} \frac{\sqrt{(2n+2)(2n+3)}}{(n!)(n+1)!} \left[\frac{\tanh r}{2} \right]^{2n}. \tag{25b}$$

From equations (22) and (24a, 24b) it is obvious that $V(X_1)$ is always > 0 and hence this quadrature can not show any squeezing in neither NLVSS nor NLESS. Now $V(X_1)$ and $V(X_2)$, which represent the NQS for the NLVSS are calculated and the results are presented in Figure 2. From this figure it is seen that squeezing in the quadrature $V(X_2)$ increases by increasing the squeezing parameter r. Numerical calculations of the NQS of the NLESS show that squeezing does not occur in either quadrature. This implies that only the NLVSS exhibits quadrature squeezing.

To look at the rotated quadratures, we define

$$\hat{X}_1(\theta) = \frac{1}{2}(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}) \text{ and } \hat{X}_2(\theta) = \frac{1}{2i}(\hat{a}e^{i\theta} - \hat{a}^\dagger e^{-i\theta}).$$

Taking into account the results (24, 25), we reach the following

$$V(X_j(\theta)) = V(X_j) + 2\langle \hat{a}^2 \rangle \sin^2\theta, \quad j = 1, 2.$$

This shows that the fluctuations in the rotated quadrature are larger than the non rotated. This means that no normal squeezing appears for these states.

4.3 Amplitude squared squeezing (ASS)

Now we use the concept of ASS introduced by Hillery [47]. This type of squeezing arises in a natural way in second-harmonic generation and in a number of non-linear optical processes.

When we study ASS for any state, our problem may be treated by introducing the field quadrature operators

$$\hat{Y}_0 = \frac{1}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}), \tag{26a}$$

$$\hat{Y}_1 = \frac{1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2}), \tag{26b}$$

$$\hat{Y}_2 = \frac{1}{4}(\hat{a}^2 - \hat{a}^{\dagger 2}). \tag{26c}$$

Operators $\hat{Y}_1$ and $\hat{Y}_2$ satisfy the commutation relation

$$[\hat{Y}_1, \hat{Y}_2] = i\hat{Y}_0. \tag{27}$$

So that the uncertainty principle applied to $\hat{Y}_1$ and $\hat{Y}_2$ is

$$(\Delta\hat{Y}_1)^2(\Delta\hat{Y}_2)^2 \geq \frac{1}{4}|\langle \hat{Y}_0 \rangle|^2. \tag{28}$$

ASS holds if

$$S(\hat{Y}_1) = \mathrm{Re}\langle \hat{a}^4 \rangle + \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle - 2(\mathrm{Re}\langle \hat{a}^2 \rangle)^2 < 0, \tag{29a}$$

$$S(\hat{Y}_2) = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle - \mathrm{Re}\langle \hat{a}^4 \rangle + 2(\mathrm{Im}\langle \hat{a} \rangle)^2 < 0. \tag{29b}$$

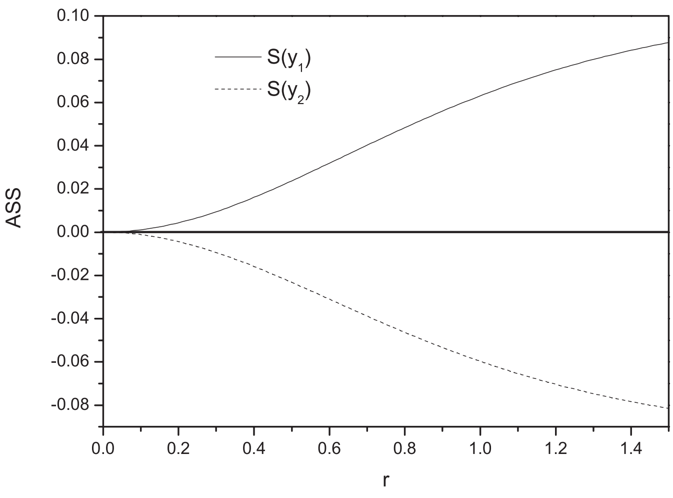
we can calculate the expectation value for $\hat{a}^4$ of the NLVSS and get

$$\langle \hat{a}^4 \rangle = |N|^2 \left[\frac{\tanh r}{2} \right]^2 \times \sum_{n=0}^{\infty} \frac{\sqrt{(2n+4)(2n+3)(2n+2)(2n+1)}}{(n!)(n+2)!} \left[\frac{\tanh r}{2} \right]^{2n}, \tag{30a}$$

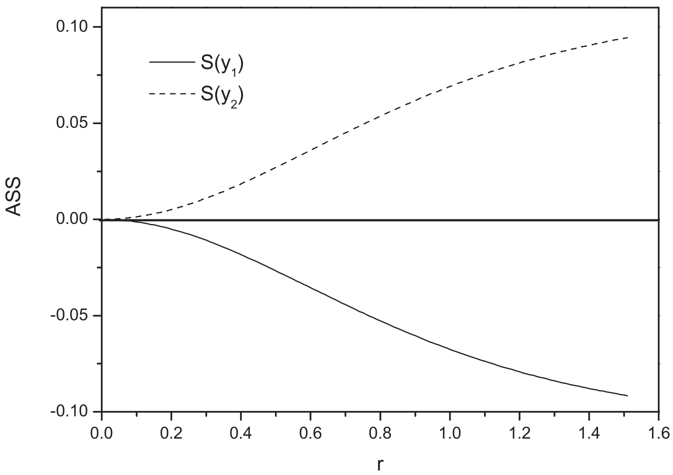
and the expectation value for $\hat{a}^4$ of the NLESS as

$$\langle \hat{a}^4 \rangle = |N|^2 \left[\frac{\tanh r}{2} \right]^2 \times \sum_{n=0}^{\infty} \frac{\sqrt{(2n+5)(2n+4)(2n+3)(2n+2)}}{(n!)(n+2)!} \left[\frac{\tanh r}{2} \right]^{2n}. \tag{30b}$$

Now $S(Y_1)$ and $S(Y_2)$, which represent the ASS are calculated and the results are presented in Figure 3. From Figure 3a it is seen that $S(Y_2)$ for the NLVSS is less than zero while $S(Y_1)$ is greater than zero for $r > 0$. Also, in Figure 3b it is seen that $S(Y_1)$ for the NLESS is less than zero while $S(Y_2)$ is greater than zero for a wide range of r. These imply that the states exhibit ASS. However the NLVSS exhibits ASS in the $S(Y_2)$ while NLESS exhibits ASS in the $S(Y_1)$.



(a)



(b)

Fig. 3. Amplitude squared squeezing for NLVSS (a) and for NLESS (b).

4.4 Phase distribution

The notion of the phase in quantum optics has found renewed interest because of the existence of phase-dependent quantum noise. In this section, the phase properties using the Pegg-Barnett method [48,49] are studied. This method is based on the phase states $|\Theta_m\rangle$, which are defined as

$$|\Theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{31}$$

where

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}; \quad m = 0, 1, ..., s. \tag{32}$$

The value of θ_0 is arbitrary. The set $\{|\Theta_m\rangle\}$ indicates a specific bases set of $(s+1)$ mutually orthogonal phase states. In fact the phase states $|\Theta_m\rangle$ are eigenstates of the Hermitian phase operator $\hat{\Phi}_\theta$ given by

$$\hat{\Phi}_\theta = \sum_{m=0}^{s} \theta_m |\Theta_m\rangle\langle\Theta_m|. \tag{33}$$

The expectation values are calculated in the finite dimensional space and after that the limit $s \to \infty$ is taken. The state of the form

$$|b\rangle = \sum_{m=0}^{s} b_n e^{in\Psi}|n\rangle \tag{34}$$

is called a partial phase state [48], where b_n are real and positive and Ψ is a phase. From equations (31) and (34), one can calculate the expectation values of the phase operator and its moments. However, we shall concentrate on the phase probability distribution. This distribution for the partial phase state of equation (34) is given by

$$P(\theta) = |\langle\Theta_m|b\rangle|^2. \tag{35}$$

Since the density of phase states is $(s+1)/2\pi$, thus in the continuum limit $s \to \infty$, equation (35) reduces to

$$P(\theta) = \frac{1}{2\pi}\left[1 + 2\sum_{n>m}\sum_{m} b_m b_n \cos[(n-m)\theta]\right]. \tag{36}$$

In what follows one calculates the phase distribution function for the present states given by equations (11) or (13). It is found to be the same for both states and is given by

$$P(r,\theta) = \frac{1}{2\pi}\left(1 + 2|N|^2 \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \frac{1}{n!m!}\left[\frac{\tanh r}{2}\right]^{(n+m)} \times \cos[2(n-m)\theta]\right). \tag{37}$$

In Figure 4, $P(r,\theta)$ as the Pegg-Barnett phase distribution given by equation (37) is plotted against the parameter r. At the squeezing parameter $r = 0$, then $P(r,\theta) = 1/2\pi$, and the phase is lost because only the vacuum state or the first excited state is presents the state (11) or (13). But at r increases, the phase starts to build up, and the information about the phase can be attained as a two-peak structure with peaks at $\theta = \pm\pi/2$.

4.5 Quasi-probability distribution functions

It has been shown from earlier studies [50–53] that the quasi-probability (Wigner-Moyal W-, Husimi-Kano Q- and Glauber-Sudarshan P-) function, are important for the statistical description of a microscopic system and provide insight into the non-classical features of the radiation fields. In this section, we concentrate on the W-functions only.

One can introduce the symmetrically ordered $C_S(\lambda)$ and anti-normally ordered $C_A(\lambda)$ characteristic function through the definitions

$$C_A(\lambda) = \mathrm{Tr}\left(\hat{\rho} e^{-\lambda^*\hat{a}} e^{\lambda\hat{a}^\dagger}\right), \quad C_S(\lambda) = \mathrm{Tr}\left(\hat{\rho} e^{\left(\lambda\hat{a}^\dagger - \lambda^*\hat{a}\right)}\right) \tag{38a}$$

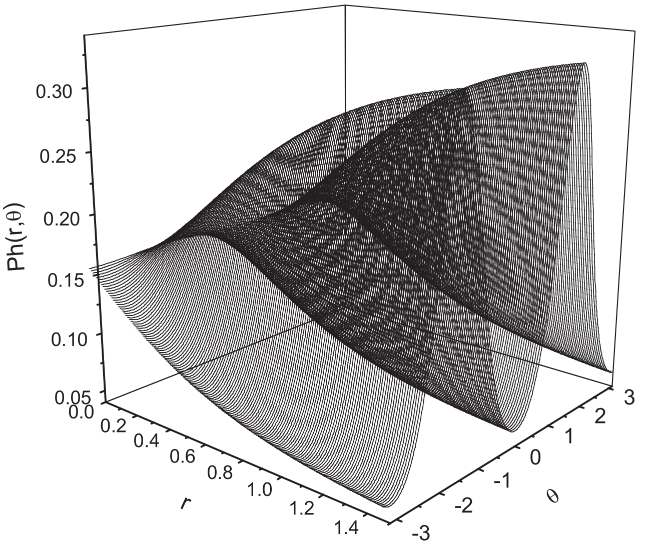


Fig. 4. The phase distribution function for NLVSS.

where $\hat{\rho}$ is the density operator and λ a complex number. It is easy to prove that [53,54]

$$C_S(\lambda) = C_A e^{\frac{1}{2}|\lambda|^2}. \tag{38b}$$

The Q-function and W-function are the Fourier transforms of the characteristic function $C_A(\lambda)$ and $C_S(\lambda)$ respectively, i.e.

$$\begin{aligned} C_A(\lambda) &= \frac{1}{\pi}\int (\langle\alpha|\hat{\rho}e^{-\lambda^*\hat{a}}e^{\lambda\hat{a}^\dagger}|\alpha\rangle)\, d^2\alpha \\ &= \int Q(\alpha)e^{(\lambda\alpha^*-\lambda^*\alpha)}\, d^2\alpha \end{aligned} \tag{39a}$$

where

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{40}$$

and

$$\begin{aligned} C_S(\lambda) &= \frac{1}{\pi}\int (\langle\alpha|\hat{\rho}e^{\lambda\hat{a}^\dagger-\lambda^*\hat{a}}|\alpha\rangle)\, d^2\alpha \\ &= \int W(\alpha)e^{(\lambda\alpha^*-\lambda^*\alpha)}\, d^2\alpha \end{aligned} \tag{41a}$$

with

$$W(\alpha) = \frac{1}{\pi^2}\int C_S(\lambda)e^{(\alpha\lambda^*-\lambda\alpha^*)}\, d^2\lambda. \tag{41b}$$

The W-function can be expressed in terms of expectation values over displaced Fock states namely [52]

$$W(\alpha) = \frac{2}{\pi}\exp^{-|\alpha|^2}\sum_{l=0}^{\infty}(-1)^l\langle\alpha, l|\hat{\rho}|\alpha, l\rangle \tag{42}$$

where the displaced Fock state

$$|\alpha, l\rangle = \sum_m C_m(\alpha, l)|m\rangle, \tag{43}$$

with

$$\begin{aligned} C_m(\alpha, l) &= \langle m|\alpha, l\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2}\sqrt{\frac{l!}{m!}}\alpha^{(m-l)}L_l^{(m-l)}(|\alpha|^2), \quad (m > l). \end{aligned} \tag{44}$$

Taking $\hat{\rho} = |\Psi\rangle\langle\Psi|$ of equations (11) the NLVSS, therefore the expression for the W-function is thus given by

$$\begin{aligned} W(\alpha) = \frac{2}{\pi}\exp(-2|\alpha|^2)\Bigg[&\sum_{n=0}^{\infty}\rho_{2n,2n}L_{2n}^0(4|\alpha|^2) \\ &+\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\rho_{2m+2n,2n}\sqrt{\frac{(2n)!}{(2n+2m)!}} \\ &\times L_{2n}^{2m}(4|\alpha|^2)\left[(2\alpha^*)^{(2m)}+(2\alpha)^{(2m)}\right]\Bigg] \end{aligned} \tag{45a}$$

where the associated Laguerre polynomial $L_{2n}^{2m}(4|\alpha|^2)$ is given by:

$$L_{2n}^{2m}(4|\alpha|^2) = \sum_{l=0}^{2n}(-1)^l\frac{(2n+2m)!(4|\alpha|^2)^l}{(2n-l)!(2m+l)!l!}.$$

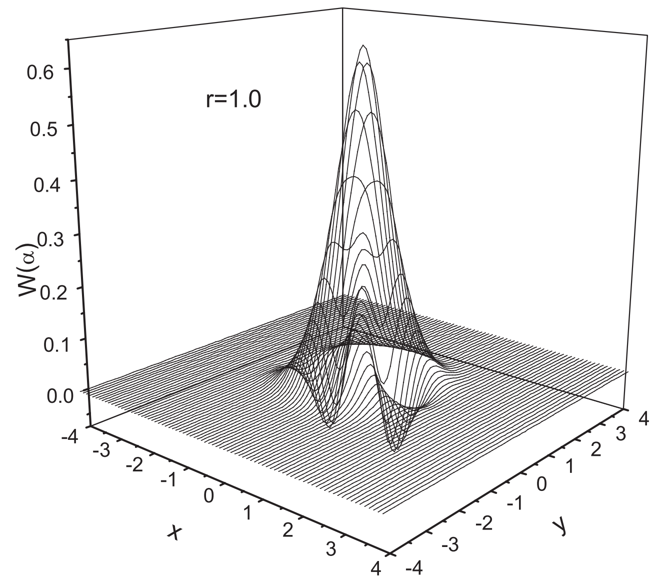
Taking $\hat{\rho} = |\Phi\rangle\langle\Phi|$ of equations (13) the NLESS, therefore the expression for the W-function is thus given by

$$\begin{aligned} W(\alpha) = -\frac{2}{\pi}\exp(-2|\alpha|^2)\Bigg[&\sum_{n=0}^{\infty}\rho_{2n+1,2n+1}L_{2n+1}^0(4|\alpha|^2) \\ &+\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\rho_{2m+2n+2,2n+1}\sqrt{\frac{(2n+1)!}{(2n+2m+2)!}} \\ &\times L_{2n+1}^{2m+1}(4|\alpha|^2)[(2\alpha^*)^{(2m+1)}+(2\alpha)^{(2m+1)}]\Bigg]. \end{aligned} \tag{45b}$$

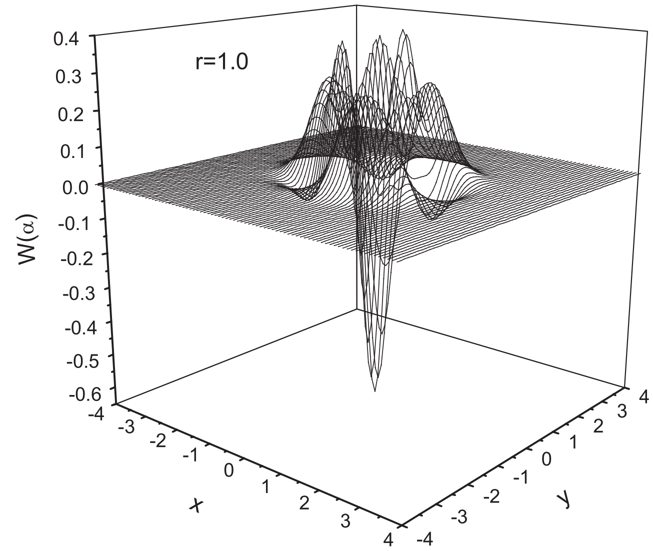
In Figure 5a, $W(\alpha)$ according to (45a) for NLVSS is plotted against the parameter $\alpha = x + iy$. At $r = 0$ the state reduces to the vacuum state and the figure shows the usual Wigner function for the vacuum state which is always positive and Gaussian. As r increases, Figure 5a we note that interference pattern as well as squeezing are exhibited. Furthermore, negative values are attained by the Wigner function for the nonlinear squeezed state which is not present for the squeezed vacuum state. In Figure 5b, $W(\alpha)$ according to (45b) for NLESS is plotted against the parameter $\alpha = x+iy$. The state $|1\rangle$ is present when $r = 0$ and hence its usual function is the result when one takes $r = 0$. In Figure 5b, as r increases the negative value at the origin with some peaks surrounding this dip are shown. Effect of squeezing can be observed in the asymmetry around the x- and y-axes.

4.6 Position distribution

Now let us investigate the position distribution $P(x, r)$, which can be measured in homodyne processes [55]. The



(a)



(b)

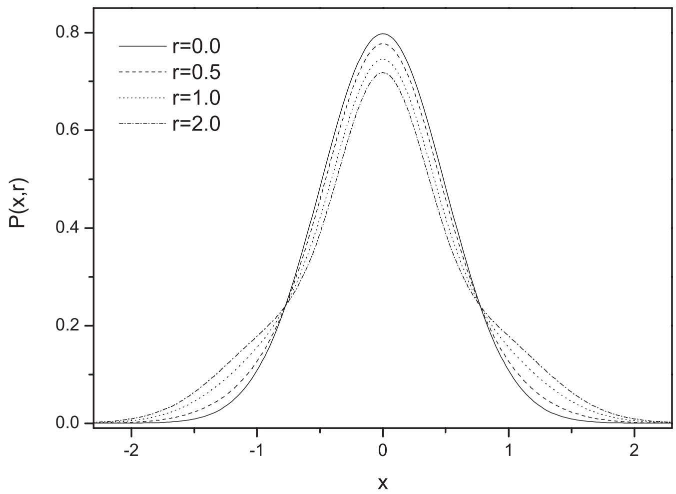
Fig. 5. Wigner function for NLVSS (a) and for NLESS (b).

position distribution can be evaluated via the Wigner function through the relation

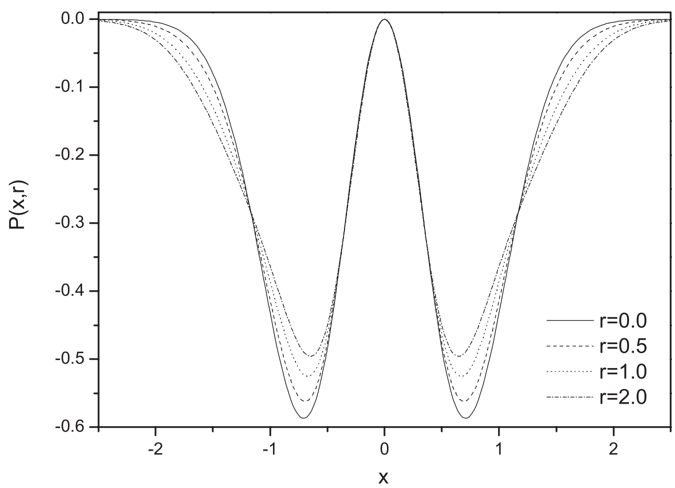
$$P(x,r) = \int_{-\infty}^{\infty} W(x+iy,r)dy. \tag{46}$$

Therefore if one uses equations (45), then the position distribution for the NLVSS becomes

$$P(x,r) = \sqrt{\frac{2}{\pi}}\exp(-2x^2)|N|^2 \sum_{n=0}^{\infty} \frac{1}{2^{(2n)}(2n)!(n!)^2} \times \left[\frac{\tanh r}{2}\right]^{2n} \left[H_{2n}(\sqrt{2}x)\right]^2, \tag{47a}$$



(a)



(b)

Fig. 6. Position distribution for NLVSS (a) and for NLESS (b).

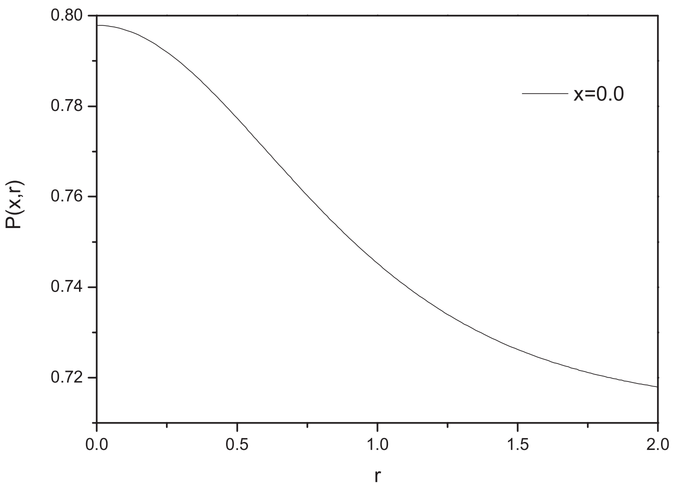
and for the NLESS becomes

$$P(x,r) = -\sqrt{\frac{2}{\pi}}\exp(-2x^2)|N|^2 \sum_{n=0}^{\infty} \frac{1}{2^{(2n+1)}(2n+1)!(n!)^2} \times \left[\frac{\tanh r}{2}\right]^{2n} \left[H_{2n+1}(\sqrt{2}x)\right]^2, \tag{47b}$$

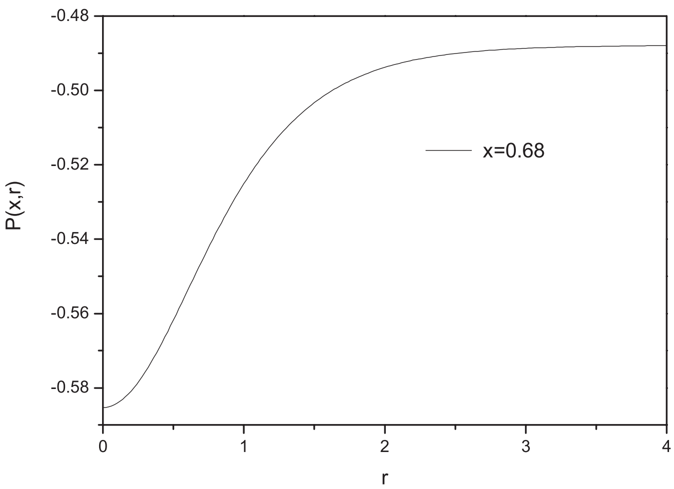
where $H_m(z)$ is the Hermite polynomial of degree n which is given by

$$H_m(z) = \sum_{s=0}^{[m/2]} (-1)^s \frac{m!(2z)^{(m-2s)}}{s!(m-2s)!}. \tag{48}$$

In Figure 6a, $P(x,r)$ of equation (47a) is plotted against the variable x for different values of r. In this figure, $P(x,r)$ is always positive with almost Gaussian shape. The maximum for $P(x,r)$ is attained at $x = 0$ and its value decreases by increasing r. It is observed that at $x \simeq \pm 0.75$ the curves have the same values of ($\simeq$0.25). In



(a)



(b)

Fig. 7. Position distribution for NLVSS (a) and for LNVSS (b).

Figure 6b, $P(x,r)$ of equation (47b) is plotted against the variable x for different values of r. In this figure, $P(x,r)$ attains negative values, it starts at $x = 0$ with the value $P(0) = 0$ for all r. As x moves away from the center, the value of P decreases to a minimum. After that it increases to reach the value zero as $|x| > 3$. For the values of r depicted, we find that all curves meet at $|x| = 1.13$ at the value $P(\pm 1.13, r) = -0.3$. In Figure 7a, $P(0,r)$ of equation (47a) is plotted against the variable r. As can be seen the maximum values decreases as r increases but it settles at a value ($\simeq$0.71) for $r > 3$. In Figure 7b, $P(0.68, r)$ of (47b), is plotted against the variable r. In this figure, the minimum value for $P(0.68, r)$ exhibits increase as r increases. It saturates at $r \simeq 3.76$ at the value $P(0.68, 3.76) = -0.487$.

5 Conclusion

A class of NLVSS and NLESS has been presented. Generation schemes have been mentioned. The states have been constructed and the case of $f(\hat{n}) = \sqrt{\hat{n}}$ has been chosen as the nonlinearity function. The effect of the nonlinearity function on the different characteristics of the states has been discussed. The autocorrelation function has been discussed and sub-Poissonian distribution has been shown to exist for certain choice of the squeezing parameter. Squeezing phenomena have been discussed and shown to exist for these states. The phase distribution has been computed and two peaks have been exhibited. The Wigner function has been calculated and investigated for such class. Also the position distribution has been calculated and the saturation value of the squeezing parameter has been determined. An earlier investigation [41] took the non-linearity function describing the center-of-mass motion of trapped ions. Further investigations concerning the form of the non-linearity function may be carried out to show the rich properties of these states. Comparisons between the vacuum and the first order excited classes are marked. Finally, these states may find applications in related fields of quantum optics and quantum information as mentioned in the introduction.

The author would like to thank the referees for their constrictive comments that helped to improve the text in many ways.

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